ANSWERS

1. A
2. B
3. C
4. B
5. C
6. E (11/2)
7. A
8. A
9. D
10. D
11. E (I, the identity matrix)
12. D
13. D
14. C
15. B
16. B
17. E (14)
18. A
19. D
20. D
21. D
22. C
23. B
24. B
25. E (7282)
26. B
27. D
28. A
29. B
30. B

SOLUTIONS

1. We subtract 2 from both sides to get

$$\frac{3x-5}{x+2}-2=\frac{3x-5-2x-4}{x+2}=\frac{x-9}{x+2}\leq 0.$$

This fraction is negative on the interval $(-2,9]$. Hence, $p+q=-2+9=7$.

2. We write the complex number in polar form and use DeMoivre's Theorem to find the fourth roots. Since

$$\sqrt{4^2+\left(4\sqrt{3}\right)^2}=\sqrt{16+48}=\sqrt{64}=8$$

and

$$\tan^{-1}\left(\frac{-4\sqrt{3}}{4}\right)=\tan^{-1}\left(-\sqrt{3}\right)=300°,$$

we have

$$\left(4-4i\sqrt{3}\right)^{1/4}=\left(8e^{300°i}\right)^{1/4}=\sqrt[4]{8}e^{75°i}$$

so that all the fourth roots occur at 90° intervals starting from 75°. Thus the angles are 75°, 165°, 255°, and 345°. Of the answers choices, the only one which is a fourth root is the one at 165°.

3. A vector orthogonal to two given vectors is the cross product of the two given vectors. Hence,

$$\vec{w}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 1 & 2 & 2\\ 7 & 0 & 24\end{vmatrix}=(2\cdot 24-0\cdot 2)\vec{\imath}-(1\cdot 24-7\cdot 2)\vec{\jmath}+(1\cdot 0-7\cdot 2)\vec{k}=48\vec{\imath}-10\vec{\jmath}-14\vec{k}.$$

The magnitude of this vector is

$$\sqrt{48^2+10^2+14^2}=2\sqrt{24^2+5^2+7^2}=2\sqrt{576+25+49}=2\sqrt{650}=10\sqrt{26}.$$

4. We complete the square on each ellipse in the answer choices, and write the equations in standard form.

A) $x^2+10y^2-6x-80y+159=0$
B) $3x^2+7y^2-42x+56y+238=0$
C) $x^2+3y^2-4x-24y+46=0$
D) $9x^2+10y^2-18x-80y+79=0$

$\Rightarrow$

A) $(x-3)^2+10(y-4)^2=10$
B) $3(x-7)^2+7(y-4)^2=21$
C) $(x-2)^2+3(y-4)^2=6$
D) $9(x-1)^2+10(y-4)^2=90$

$\Rightarrow$

A) $\frac{(x-3)^2}{10}+(y-4)^2=1$
B) $\frac{(x-7)^2}{7}+\frac{(y-4)^2}{3}=1$
C) $\frac{(x-2)^2}{6}+\frac{(y-4)^2}{2}=1$
D) $\frac{(x-1)^2}{10}+\frac{(y-4)^2}{9}=1$

Note that each ellipse has a horizontal major axis. In an ellipse, $a^2-b^2=c^2$, where c is the distance from the center to the focus. Checking this relationship reveals that in answer choice B, we have $7-3=4=c^2$ so that $c=\pm 2$, but a horizontal distance of 2 away from the center $(7,4)$ means that the foci are at $(5,4)$ and $(9,4)$, not $(0,4)$.

5. Addition of the two matrices is not defined since they are not the same size. Multiplication is defined since the rows of one matrix is equal in number to the columns of the other; hence, multiplication is defined in both orders. However, the product of a 2×3 and a 3×2 gives a 2×2, and the product of a 3×2 and a 2×3 gives a 3×3, so $AB - BA$ is not defined. Therefore, two of the four operations are defined.

6. There are many ways to do this problem. The quickest is probably using the fact that the area of the triangle is half the magnitude of the cross product of the vectors $\langle 5, 3, 4\rangle$ and $\langle 3, 0, 2\rangle$. Thus,

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 5 & 3 & 4 \\ 3 & 0 & 2 \end{vmatrix} = (3 \cdot 2 - 0 \cdot 4)\vec{\imath} - (5 \cdot 2 - 3 \cdot 4)\vec{\jmath} + (5 \cdot 0 - 3 \cdot 3)\vec{k} = 6\vec{\imath} + 2\vec{\jmath} - 9\vec{k}$$

is the cross product. Half of the magnitude is therefore

$$\frac{1}{2}\sqrt{6^2 + 2^2 + 9^2} = \frac{1}{2}\sqrt{36 + 4 + 81} = \frac{1}{2}\sqrt{121} = \frac{11}{2}.$$

7. First, note that when $x = 2$, we have $4 + 4y - 3y = 3$, which implies $y = -1$. The derivative of the implicit function $F = x^2 + 2xy - 3y - 3$ is

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x + 2y}{2x - 3}.$$

Evaluated at $(2, -1)$, we get $\frac{dy}{dx} = -\frac{4-2}{4-3} = -2$.

8. The denominator of our probability fraction is the length of the interval: $2/3 - (-5/7) = 2/3 + 5/7 = 29/21$. The numerator is the length of the subinterval of $[-5/7, 2/3]$ where

$$kx^2 - \frac{x\sqrt{k}}{2} + \frac{k}{4} = 0$$

has two real roots. This equation has two real roots when the discriminant is positive; that is, when

$$b^2 - 4ac = \left(-\frac{\sqrt{k}}{2}\right)^2 - 4k\left(\frac{k}{4}\right) = \frac{k}{4} - k^2 > 0.$$

This inequality can be written as $k - 4k^2 = k(1 - 4k) > 0$. This inequality is true when k and $1 - 4k$ have the same sign; this happens only for $0 < k < 1/4$. The length of this subinterval is $1/4$. Therefore, the probability fraction must be $(1/4)/(29/21) = 21/116$.

9. Note that the mean is $(2 + 2 + 2 + 4 + 5 + 10 + n)/7 = (25 + n)/7$. Now, n cannot be 2 or less than 2, for if it were, then the mode and the median would both be 2, implying the arithmetic progression is constant which it cannot be. So, we must have $n \geq 3$. If $n = 3$, then the mode is 2, the median is 3, and the mean is $(25 + 3)/7 = 4$. Since 2, 3, and 4 are in non-constant arithmetic progression, $n = 3$ works. Since the mode and the median must be integers, the mean must also be an integer. Thus, we try values of $n > 3$ such that $(25 + n)/7$ will be an integer, and we note that if $n > 3$, then the median must be 4. For $n = 10$, the mean is 5, and the sequence is 2, 4, 5, which is not in arithmetic progression. For $n = 17$, the mean is 6, and the sequence is 2, 4, 6, which is arithmetic; hence, $n = 17$ works. For any value of $n > 17$ which makes the mean an integer, the mean will be the last number in the sequence and it will be greater than 6, so the sequence cannot be arithmetic. Therefore, the sum of the values of n is $3 + 17 = 20$.

10. Clearly, $\log(10^6) = 6$ and $\log(10^7) = 7$, and each integer given is between 10^6 and 10^7. Hence, the logarithm of each number is between 6 and 7. The logarithm of a product is the sum of the logarithms, so $\log P$ is between 24 and 28. Now let's narrow this range down further. We have

$$\log(4 \cdot 10^6) = 6 + 2\log 2 \approx 6 + 2 \cdot 0.3 = 6.6$$

and

$$\log(8 \cdot 10^6) = 6 + 3\log 2 \approx 6 + 3 \cdot 0.3 = 6.9.$$

Thus $\log P$ is between 26.4 and 27.6. As the numbers in the product average a bit more than 6 million, we expect $\log P$ to be a bit more than the average of 26.4 and 27.6. Hence, the closest integer is 27. (Indeed, the actual value of $\log P$ is 27.118, rounded to the thousandths.)

11. Multiplying the given equation by A on the right and by A^{-1} on the left gives $A^{-1}ABCA = A^{-1}IA$. This simplifies to $IBCA = A^{-1}A$, which becomes $BCA = I$.

12. Letting $x = 0$ results in the indeterminate form $0/0$, so we use l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{\ln 3}{\ln 2} = \log_2 3.$$

13. Notice that the numerator is larger than the denominator. Thus, we can simplify this fraction into a whole part and a fractional part:

$$\frac{N + 314}{N + 123} = \frac{N + 123 + 314 - 123}{N + 123} = 1 + \frac{191}{N + 123}.$$

This means that the fraction $191/(N + 123)$ must be an integer, which implies that $N + 123$ must be a factor of 191. Since 191 is prime, we see that $N + 123$ must be equal to either 1 or 191. Since $N + 123 = 1$ results in a negative solution, we set $N + 123 = 191$ to get $N = 68$. (When $N = 68$, the fraction becomes 2.) The sum of the digits of N is 14.

14. The medians intersect at the centroid, and the centroid splits each median into the ratio $2:1$. The medians also split the triangle into six triangles of equal area. We find one of these areas, and multiply by 6. Since two of the given medians are equal, the triangle must be isosceles, so that the median of length 96 is an altitude of the triangle. Then there exists a right triangle with hypotenuse of the longer part of a median of length 60, one leg half the base of the triangle, and the other leg as the shorter part of the median of length 96. Hence, the hypotenuse has length $60 \cdot 2/3 = 40$ and one leg has length $96/3 = 32$. Then the other leg has length $\sqrt{40^2 - 32^2} = \sqrt{1600 - 1024} = \sqrt{576} = 24$. Finally, the area of this right triangle is $24 \cdot 32/2 = 384$ so the area of the entire triangle is $6 \cdot 384 = 2304$.

15. Let B, G, and Y denote the number of blue, green, and yellow cubes, respectively. We are told in the problem that $G \geq B/2$, $G \leq Y/3$, and $B + G \geq 34$. Combining the first two inequalities gives us $3B \leq 6G \leq 2Y$. So to find the minimum value of Y requires us to find the minimum value of $6G$, and therefore the minimum value of $3B$. But we must choose a value of B which satisfies both $G \geq B/2$ and $B + G \geq 34$. Since G must be at least half of B, this is an approximate ratio of $G{:}B = 1{:}2$. Since $B + G \geq 34$, the closest values to this ratio are either $(B, G) = (23, 11)$ or $(B, G) = (22, 12)$. However $(B, G) = (23, 11)$ does not satisfy the inequality $G \geq B/2$. Using $(B, G) = (22, 12)$ gives $Y = 36$. These values satisfy all three inequalities in the problem.

16. If one distributes and then combines like terms on the left side of the equation, one will obtain the term $(a+b+c)x^2$. However, there is no x^2 term on the right side of the equation. Hence $a+b+c=0$.

17. We use the Euclidean algorithm. We obtain

$$\begin{aligned} 4897 &= 1357\cdot 3+826 \\ 1357 &= 826\cdot 1+531 \\ 826 &= 531\cdot 1+295 \\ 531 &= 295\cdot 1+236 \\ 295 &= 236\cdot 1+59 \\ 236 &= 59\cdot 4 \end{aligned}$$

The last nonzero remainder is the greatest common factor; this is 59. The sum of the digits is 14.

18. Distributing, we get $2^{101}+2^{51}+2^{50}+1$. Thus, the binary representation will require 102 digits and four of them will be 1. Therefore, there are $102-4=98$ zeroes.

19. We use the fact that $\log 2\approx 0.3$ and $\log 3\approx 0.5$. Then we can approximate $3^{31}\approx 10^{(0.5)(31)}=10^{15.5}$, $2^{31}\approx 10^{(0.3)(31)}=10^{9.3}$, $3^{29}\approx 10^{(0.5)(29)}=10^{14.5}$, and $2^{29}\approx 10^{(0.3)(2.9)}=10^{8.7}$. Hence,

$$\left\lfloor\frac{3^{31}+2^{31}}{3^{29}+2^{29}}\right\rfloor=\left\lfloor\frac{3^{31}\left(1+\frac{2^{31}}{3^{31}}\right)}{3^{29}\left(1+\frac{2^{29}}{3^{29}}\right)}\right\rfloor=\left\lfloor 9\left(\frac{1+\frac{10^{9.3}}{10^{15.5}}}{1+\frac{10^{8.7}}{10^{14.5}}}\right)\right\rfloor=\left\lfloor 9\left(\frac{1+10^{-6.2}}{1+10^{-5.8}}\right)\right\rfloor.$$

Now, when x is small, $\frac{1}{1+x}\approx 1-x$. Here, x is very small! Thus,

$$\left\lfloor\frac{3^{31}+2^{31}}{3^{29}+2^{29}}\right\rfloor=\left\lfloor 9(1+10^{-6.2}-10^{-5.8})\right\rfloor=8$$

because the quantity in parentheses is slightly less than 1. (Indeed, the fraction itself is approximately 8.99996.)

20. The possible prime sums one can obtain by rolling three dice are 3, 5, 7, 11, 13, and 17. There is only 1 way to roll a 3: 1-1-1. There are 6 ways to roll a 5: 1-1-3 (3 ways) and 1-2-2 (3 ways). There are 15 ways to roll a 7: 1-1-5 (3 ways), 1-2-4 (6 ways), 1-3-3 (3 ways), and 2-2-3 (3 ways). There are 27 ways to roll 11: 1-4-6 (6 ways), 1-5-5 (3 ways), 2-3-6 (6 ways), 2-4-5 (6 ways), 3-3-5 (3 ways), and 3-4-4 (3 ways). There are 21 ways to roll 13: 1-6-6 (3 ways), 2-5-6 (6 ways), 3-5-5 (3 ways), 3-4-6 (6 ways), and 4-4-5 (3 ways). There are 3 ways to roll 17: 5-6-6. The total number of ways to get a prime sum is $1+6+15+27+21+3=73$. Of these, there are $1+3+3+3+6+3+6+3+3=31$ ways in which the smallest number rolled is a 1. Hence, the probability is $31/73$.

21. There are many ways to do this problem, but the method we present relies on the fact that the constant terms don't matter as x approaches infinity. So we will simply change 9 to $9/4$ and 16 to $25/4$. Then we can easily compute the limit. We have

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x+\frac{9}{4}}-\sqrt{x^2-5x+\frac{25}{4}}\right)=\lim_{x\to\infty}\left(\sqrt{\left(x+\frac{3}{2}\right)^2}-\sqrt{\left(x-\frac{5}{2}\right)^2}\right)=\lim_{x\to\infty}\left(x+\frac{3}{2}-x+\frac{5}{2}\right),$$

which equals 4.

22. We use complementary counting: we will subtract from all possible paths from the origin to $(3,3,3)$ the number of such paths that go through the point $(2,2,2)$. Every path from the origin to $(3,3,3)$ will contain 9 unit lattice segments, 3 in each axial direction. Hence, the number of ways is

$$\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{9\cdot8\cdot7}{3\cdot2}\cdot\frac{6\cdot5\cdot4}{3\cdot2}\cdot1 = 3\cdot4\cdot7\cdot5\cdot4 = 1680.$$

The number of paths that go through $(2,2,2)$ is the same as the number of paths from the origin to $(2,2,2)$, multiplied by the number from $(2,2,2)$ to $(3,3,3)$. This is

$$\binom{6}{2}\binom{4}{2}\binom{2}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{6\cdot5}{2}\cdot\frac{4\cdot3}{2}\cdot1\cdot3\cdot2\cdot1 = 3\cdot5\cdot2\cdot3\cdot3\cdot2 = 540.$$

Finally, the number of paths that do not go through $(2,2,2)$ is $1680-540=1140$.

23. To make this a little simpler, let's call $a=\log(x)$, $b=\log(y)$, and $c=\log(z)$. Then, using properties of logarithms, we can rewrite the system as

$$\begin{array}{c} \log(3xy) = \log(x)\log(y) \\ \log(yz) = \log(y)\log(z) \\ \log(3xz) = \log(x)\log(z) \end{array} \quad\Longrightarrow\quad \begin{array}{c} \log 3 + a + b = ab \\ b + c = bc \\ \log 3 + a + c = ac \end{array}$$

Using Simon's Favorite Factoring Trick, we can factor the system of equations.

$$\begin{array}{c} \log 3 + a + b = ab \\ b + c = bc \\ \log 3 + a + c = ac \end{array} \quad\Longrightarrow\quad \begin{array}{c} ab - a - b + 1 = 1 + \log 3 \\ bc - b - c + 1 = 1 \\ ac - a - c + 1 = 1 + \log 3 \end{array} \quad\Longrightarrow\quad \begin{array}{c} (a-1)(b-1) = \log 30 \\ (b-1)(c-1) = 1 \\ (a-1)(c-1) = \log 30 \end{array}$$

Now, multiply the first and third equations and divide by the second. We get $(a-1)^2 = (\log 30)^2$. Hence, $a = \log 30 + 1 = \log 300$. Now, solving for b in the first equation, we get

$$b = \frac{\log 30}{a-1} + 1 = \frac{\log 30}{\log 30 + 1 - 1} + 1 = 2.$$

Due to the symmetry of the third equation, we must have $c = 2$ as well. Hence, $x = 300$, $y = 100$, and $z = 100$. Thus, $x + y + z = 500$.

24. Extend $\overline{AB}$ and $\overline{DC}$ to P. Since $\measuredangle ABC + \measuredangle BCD = 270°$, then $\measuredangle CDA + \measuredangle DAB = 90°$. It follows that $\angle P$ is right, and thus APD and BPC are right triangles. Hoping that $\angle BPC$ has integral side lengths and recalling the 8-15-17 Pythagorean triple, we try $BP = 15$ and $CP = 8$. If these lengths are the right ones, $AP = 20$ and $DP = 15$, which is consistent with $\angle APD$ being right. The area, therefore, is the difference between the areas of triangle APD and triangle BPC, namely, $20\cdot15/2 - 15\cdot8/2 = 90$.

25. Every positive integer with two proper divisors is of the form p^2, where p is prime. If both of these factors are to be less than 30, p must be less than 30; therefore, the information we are given is that $2^2+3^2+5^2+\cdots+29^2 = 2397$. Every positive integer with three proper divisors must be of the form p^3 where p is prime or of the form pq where p and q are prime. If both of these factors are to be less than 30, either p^2 must be less than 30 in the first case, or both p and q must be less than 30 in the second case. For the first case, the perfect cubes that satisfy the given conditions are 8, 27, and 125, the sum of which is 160. For the second case, consider the expansion of $(2+3+5+\cdots+29)^2$. Said expansion will be the sum of twice the sum of the desired integers of the form pq and the integers of the form p^2. In other words, the desired sum for the second case is

$$\frac{(2+3+5+\cdots+29)^2 - 2397}{2} = \frac{129^2 - 2397}{2} = \frac{16641 - 2397}{2} = 7122.$$

It follows that the final answer is $160 + 7122 = 7282$.

26. Notice that $\frac{1}{2}[f(1) + f(-1)] - f(0)$ is always eqaul to d. At minimum, Polya must ask about 3 values of x.

27. Rewrite n as $5^{2017} + 5^{2016} + 5^{2015} + \cdots + 1$. It is well-known that the number of terminating zeroes in an integer x is equal to

$$\left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{5^2} \right\rfloor + \left\lfloor \frac{x}{5^3} \right\rfloor + \cdots$$

Applying this to n, we find that

$$f(n) = (5^{2016} + 5^{2015} + \cdots + 1) + (5^{2015} + 5^{2014} + \cdots + 1) + \cdots + (5 + 1) + 1.$$

Each term in parentheses is a geometric series, so we may find each sum. We get

$$f(n) = \frac{5^{2017} - 1}{4} + \frac{5^{2016} - 1}{4} + \cdots + \frac{5^2 - 1}{4} + \frac{5^1 - 1}{4}.$$

Naturally, this simplifies.

$$f(n) = \frac{5^{2017} + 5^{2016} + \cdots + 5^2 + 5 - 2017}{4} = \frac{(5^{2018} - 5)/4 - 2017}{4} = \frac{5^{2018} - 8073}{16}$$

The answer to the problem, therefore, is $a + b + c + d = 16 + 5 + 2018 + 8073 = 10112$.

28. There are many ways to do this problem, but in an effort to make you think, I will use the following fascinating relationship, which you should attempt to prove: In any triangle with angles α, β, and γ and with circumradius R and inradius r, we have

$$\cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R}.$$

Thus, we compute

$$\frac{r}{R} = \cos 30° + \cos 60° + \cos 90° - 1 = \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 - 1 = \frac{\sqrt{3} - 1}{2}.$$

29. Note that $P(3) = 2481 = 3 \cdot 827 = 3 \cdot P(1)$ and that $P(2) = 1654 = 2 \cdot 827 = 2 \cdot P(1)$. So we form a new polynomial to take advantage of these values. Let $Q(x) = P(x) - 827x$. Then $Q(x)$ has degree 4, and we know three of the roots: $Q(1) = P(1) - 827 = 0$, $Q(2) = P(2) - 827 \cdot 2 = 0$, and $Q(3) = P(3) - 827 \cdot 3 = 0$. Hence, $Q(x) = (x - 1)(x - 2)(x - 3)(x - r)$ for some real r. Now we compute

$$\frac{1}{4}[P(5) + P(-1)] = \frac{1}{4}[Q(5) + 827 \cdot 5 + Q(-1) - 827] = \frac{1}{4}[Q(5) + Q(-1)] + 827.$$

Therefore, we can write this as

$$\frac{1}{4}[(5 - 1)(5 - 2)(5 - 3)(5 - r) + (-1 - 1)(-1 - 2)(-1 - 3)(-1 - r)] + 827.$$

Certainly, this simplifies, so we have

$$\frac{1}{4}[P(5) + P(-1)] = \frac{1}{4}[4 \cdot 3 \cdot 2(5 - r) + 2 \cdot 3 \cdot 4(1 + r)] + 827 = 30 - 6r + 6 + 6r + 827 = 863.$$

30. Clearly, $f(2) = 1$ since this shuffle simply puts the 2-card deck back in the same order. We were given $f(4) = 2$. Now we compute $f(6)$. Let the cards be numbered 1 through 6. Then we track the shuffles, where each row below is a shuffle.

		A	B
Original	1 2 3 4 5 6	1 2 3	4 5 6
1st shuffle	1 4 2 5 3 6	1 4 2	5 3 6
2nd shuffle	1 5 4 3 2 6	1 5 4	3 2 6
3rd shuffle	1 3 5 2 4 6	1 3 5	2 4 6
4th shuffle	1 2 3 4 5 6		

So we have $f(6) = 4$. Next, we track an 8-card deck with cards numbered 1 through 8.

		A	B
Original	1 2 3 4 5 6 7 8	1 2 3 4	5 6 7 8
1st shuffle	1 5 2 6 3 7 4 8	1 5 2 6	3 7 4 8
2nd shuffle	1 3 5 7 2 4 6 8	1 3 5 7	2 4 6 8
3rd shuffle	1 2 3 4 5 6 7 8		

So we have $f(8) = 3$. Finally, we track a 10-card deck with cards numbered 1 through 10.

		A	B
Original	1 2 3 4 5 6 7 8 9 10	1 2 3 4 5	6 7 8 9 10
1st shuffle	1 6 2 7 3 8 4 9 5 10	1 6 2 7 3	8 4 9 5 10
2nd shuffle	1 8 6 4 2 9 7 5 3 10	1 8 6 4 2	9 7 5 3 10
3rd shuffle	1 9 8 7 6 5 4 3 2 10	1 9 8 7 6	5 4 3 2 10
4th shuffle	1 5 9 4 8 3 7 2 6 10	1 5 9 4 8	3 7 2 6 10
5th shuffle	1 3 5 7 9 2 4 6 8 10	1 3 5 7 9	2 4 6 8 10
6th shuffle	1 2 3 4 5 6 7 8 9 10		

And we have $f(10) = 6$. Hence, $N = f(2) + f(4) + f(6) + f(8) + f(10) = 1 + 2 + 4 + 3 + 6 = 16$, and the sum of the digits of N is 7.

Note: As it turns out, card-shuffling has lots to do with number theory. Indeed, if $f(n) = k$ is the number of perfect out-shuffles on a deck with n cards, then k is the smallest positive integer such that 2^k is congruent to 1 modulo $n - 1$.